Unifying Analysis of Ergodic MIMO Capacity in Correlated Rayleigh Fading Environments

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***Abstract*: Recently the authors solved the open problem of calculating the moment generating function of mutual information of arbitrarily correlated MIMO channels with fading correlation at transmitter as well as receiver in a flat Rayleigh fading environment by a novel mathematical approach [19]. Based on this general result, we outline the non-trivial derivation of the moment generating function of mutual information for the special scenarios of one-side correlated and uncorrelated channels, thereby taking advantage of certain properties of the Kummer U function. Using the moment generating functions, we derive exact formulas of the ergodic capacity for these special cases, thus unifying the analysis of correlated Rayleigh fading MIMO channels. It turns out that the ergodic capacity can be expressed in terms of a sum of determinants with elements that are a combination of polynomials, exponentials, and the exponential integral E_1 solely. The results are non-asymptotic and thus hold for an arbitrary number of antenna elements at transmitter as well as receiver. The analysis is verified by Monte-Carlo simulations and shows a perfect match.**

1. Introduction

In his seminal paper [1] Telatar calculated the ergodic capacity of i.i.d. MIMO channels, predicting enormous capacity gains by using antenna arrays at both wireless transmitter and receiver. For the derivation, Telatar integrated over the eigenvalue probability density function (pdf) of certain complex Wishart matrices, resulting in capacity expressions that were given in terms of a single integral, thus allowing for a simple calculation. Later in [2], Foschini and Gans presented analytical bounds and simulation results on ergodic and outage capacity for uncorrelated channels. In [3], bounds were derived for correlated MIMO channels with fading correlation at either transmitter or receiver and a tight bound for MIMO systems with fading correlation at both the transmit and receive antenna array was given in [18]. A moment generating function (MGF) approach was used in [4], were the authors derived the MGF of mutual information of an i.i.d. MIMO channel, again by integrating over the eigenvalue pdf of complex Wishart matrices. The MGF was given in terms of a determinant with elements that are single integrals, while a numerical Laplace transform inversion technique was used to find the capacity distribution.

Eigenvalue pdfs of certain central and non-central complex Wishart matrices were utilized in [5] for calculating the MGF of mutual information of MIMO channels with Ricean fading and one-side correlated channels with Rayleigh fading. From the MGF, expressions were derived for ergodic capacity (1st moment of mutual information) and the second moment, which was deployed for Gaussian [6] and Gamma [7] approximations of the outage capacity. Similar results were presented in [8] and [9] for one-side correlated channels with Rayleigh fading. However, due to mathematical difficulties with the eigenvalue pdf approach, the analysis did not cover the general case with transmit as well as receive correlation. This gap was closed recently by exploiting advanced mathematical tools from multivariate statistics, when a concise solution on the MGF of mutual information for arbitrarily correlated MIMO channels was presented in [19] in terms of a hypergeometric function of matrix arguments. Using a representation of this MGF in terms of scalar hypergeometric functions, in this paper we outline a new approach and a general way of calculating ergodic capacity in case of a fully (transmit and receive) correlated channel, a one-side correlated channel, and finally an uncorrelated channel, thus complementing earlier results, generalizing and unifying the ergodic capacity analysis of correlated Rayleigh fading MIMO channels.

It is demonstrated that all capacity expressions can be given in terms of a sum of determinants, whereas the elements are simple polynomials, exponentials, and the only special function that occurs is the exponential integral E_1. Various Monte-Carlo simulations with varying correlation properties and variable number of transmit as well as receive antennas confirm the validity and accuracy of the analysis.

2. Signal and channel model

We consider a flat fading MIMO link modeled by

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{s} + \boldsymbol{n}, \tag{1}$$

where $\mathbf{s}$ is the T×1 TX symbol vector, i.e. there are T independent data streams (subchannels), $\mathbf{H}$ is the R×T MIMO channel matrix with correlated Rayleigh fading elements, $\mathbf{n}$ is the R×1 noise vector, and $\mathbf{y}$ is the R×1 receive vector. By R we denote the number of RX antennas and T is the number of TX antennas. In the following we assume additive Gaussian noise, where the noise covariance matrix is given by $\boldsymbol{R}_{nn} = N_0 \cdot \tilde{\boldsymbol{R}}_{nn}$ with $\mathrm{tr}(\tilde{\boldsymbol{R}}_{nn}) = R$ such that N_0 is the mean noise power per antenna element. Equivalently, the normalized signal covariance matrix is given by $\boldsymbol{R}_{ss} = N_0 \cdot \tilde{\boldsymbol{R}}_{ss}$ with $\mathrm{tr}(\tilde{\boldsymbol{R}}_{ss}) = T$ such that E_s is the mean symbol energy. Both covariances can in general be colored.

Using a widely accepted channel model (e.g. [10]), the correlated MIMO channel can be described by the matrix product

$$\boldsymbol{H} = \boldsymbol{A}^H \boldsymbol{H}_w \boldsymbol{B}, \tag{2}$$

where $\mathbf{H}_w$ is a R×T matrix of complex i.i.d. Gaussian variables of unity variance and

$$\boldsymbol{A}\boldsymbol{A}^H = \boldsymbol{R}_{RX} \qquad \boldsymbol{B}\boldsymbol{B}^H = \boldsymbol{R}_{TX}, \tag{3}$$

where $\mathbf{R}_{RX}$ and $\mathbf{R}_{TX}$ is the long-term stable (normalized) receive and transmit correlation matrix, respectively.

In the remainder of the paper, by $\mathbf{I}_n$ we denote an identity matrix of size n×n (the index can be omitted, if the size of the matrix is clear from the context), $\mathbf{0}$ is a matrix of all zeros, $\mathbf{1}$ is a matrix of all ones, $\mathrm{diag}(x_1,\ldots x_n)$ is a diagonal matrix with elements $x_1,\ldots x_n$, $|\boldsymbol{X}|$ is the determinant of the quadratic matrix $\mathbf{X}$, $[x_{ij}]$ is a matrix with element x_{ij} in row i and column j, $|x_{ij}|$ is the determinant of a matrix with element x_{ij} in row i and column j, eig($\mathbf{X}$) returns a diagonal matrix of eigenvalues of $\mathbf{X}$, $\mathbf{X}^H$ means Hermitian, x≅means 'random variable (RV) x has the same distribution as', and $E_x[\ ]$ denotes expected value with respect to RV x.

3. MIMO Mutual Information

A. General expressions

For the given system model with flat fading, it is well known [1] that the mutual information between input and output vector of the MIMO channel is given by

$$I(\boldsymbol{s},\boldsymbol{y}) = \log_2\left|\boldsymbol{I}+\gamma\cdot\tilde{\boldsymbol{R}}_{ss}\boldsymbol{H}^H\tilde{\boldsymbol{R}}_{nn}^{-1}\boldsymbol{H}\right| \quad (4)$$

with signal to noise ratio (SNR) definition $\gamma = E_s/N_0$. Then introduce the following diagonal matrices of eigenvalues for brevity

$$\boldsymbol{\Omega} = \mathrm{eig}(\tilde{\boldsymbol{R}}_{nn}^{-1}\boldsymbol{R}_{RX}) = \mathrm{diag}(\omega_1,\ldots,\omega_R) = \mathrm{diag}(\boldsymbol{\omega}) . \quad (5)$$

and

$$\boldsymbol{\Sigma} = \mathrm{eig}(\tilde{\boldsymbol{R}}_{ss}\boldsymbol{R}_{TX}) = \mathrm{diag}(\sigma_1,\ldots,\sigma_T) = \mathrm{diag}(\boldsymbol{\sigma}) . \quad (6)$$

It can be shown that

$$I(\boldsymbol{s},\boldsymbol{y}) \cong \log_2\left|\boldsymbol{I}+\gamma\cdot\boldsymbol{\Sigma}\boldsymbol{H}_w^H\boldsymbol{\Omega}\boldsymbol{H}_w\right| , \quad (7)$$

where $\boldsymbol{H}_w$ is a R×T matrix of complex i.i.d. normal distributed elements. We emphasize that in the remainder of the paper we assume R≥T without loss of generality. To this end, note that with the equality (e.g. [11])

$$|\boldsymbol{I}+\boldsymbol{XY}| = |\boldsymbol{I}+\boldsymbol{YX}| \quad (8)$$

we can also reduce the case T>R to an equivalent problem by just switching $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$, as we have

$$\begin{aligned} I(\boldsymbol{s},\boldsymbol{y}) &\cong \log_2\left|\boldsymbol{I}+\gamma\cdot\boldsymbol{\Sigma}\boldsymbol{H}_w^H\boldsymbol{\Omega}\boldsymbol{H}_w\right| \\ &= \log_2\left|\boldsymbol{I}+\gamma\cdot\boldsymbol{\Omega}\boldsymbol{H}_w\boldsymbol{\Sigma}\boldsymbol{H}_w^H\right| \end{aligned} . \quad (9)$$

4. MGF of MIMO mutual information

Starting off from the MGF of mutual information of MIMO channels with fading correlation at the transmitter as well as the receiver, we derive the MGF for channels with transmit correlation only, receive correlation solely, and finally uncorrelated channels. To this end, we have to calculate the corresponding limits for the various cases via application of L'Hospital's rule for limits of the type $0/0$.

For expressing the MGF of mutual information we need some definitions. For brevity, in the following we let $\tilde{s} = s/\ln 2$ and the complex multivariate Gamma function is given by

$$\Gamma_m(r) = \prod_{i=1}^{m}\Gamma(r-i+1) , \quad (10)$$

where Γ(x) is the standard Gamma function. The Vandermonde determinant of a diagonal m×m matrix $\mathbf{X}=\mathrm{diag}(x_1,\ldots,x_m)$ can be expressed as

$$\alpha_m(\boldsymbol{X}) = \prod_{i<j}(x_i-x_j) . \quad (11)$$

Furthermore, define the auxiliary function

$$\psi_q^{(m)}(b) = \prod_{i=1}^{m}\prod_{j=1}^{q}(b_j-i+1)^{i-1} \quad (12)$$

and Pochhammer's symbol has the meaning

$$[a]_k = a\cdot(a+1)\cdot\ldots\cdot(a+k-1) \qquad [a]_0 = 1 . \quad (13)$$

The scalar hypergeometric function $_2F_0(a_1,a_2;\ ;z)$ reads

$$_2F_0(a_1,a_2;\ ;z) = \sum_{k=0}^{\infty}[a_1]_k[a_2]_k\cdot\frac{z^k}{k!} , \quad (14)$$

while U(a,b,z) is the Kummer U function ([14], chapter 13), whereas the relation between the Kummer U function and the hypergeometric function is given in [14] paragraph 13.1

$$U(a,b,z) = z^{-a}\cdot{}_2F_0\left(a, 1+a-b;\ ;-\frac{1}{z}\right) . \quad (15)$$

We also state the integral representation for the Kummer U function, whereas [14], equation 13.2.5 reads

$$U(a,b,z) = \frac{1}{\Gamma(a)}\cdot\int_0^{\infty}e^{-zt}\cdot t^{a-1}\cdot(1+t)^{b-a-1}dt . \quad (16)$$

Furthermore, for the special case a=1, the Kummer U function can be expressed in terms of the incomplete Gamma function $\Gamma(x,z)$

$$U(1,b,z) = \int_0^{\infty}e^{-zt}\cdot(1+t)^{b-2}dt = \frac{e^z\cdot\Gamma(b-1,z)}{z^{b-1}} , \quad (17)$$

which can be used for reformulating the MGF expressions in the following paragraphs.

A. MGF of fully correlated channel

The basis for all following derivations is

Theorem 1. The MGF Z_{TR} of the MIMO channel mutual information according to (7) with T transmit and R receive antennas is given by

$$Z_{TR}(s) = {}_2\boldsymbol{F}_0^{(R)}\left(-\frac{s}{\ln 2}, R;-\boldsymbol{\Sigma},\boldsymbol{\Omega}\right) , \quad (18)$$

where $_2\boldsymbol{F}_0^{(R)}(a_1,a_2;\boldsymbol{X},\boldsymbol{Y})$ is a hypergeometric function of two complex matrix arguments [12] of maximum size R×R with parameters a_1 and a_2.

Proof: See [19].

For the following analysis we consider an alternative expression for (18) in terms of scalar hypergeometric and Kummer U functions, respectively.

Theorem 2. The MGF Z_{TR} of the fully correlated MIMO channel mutual information according to (7) with T transmit and R receive antennas is given by

$$Z_{TR}(s) = \frac{\chi}{\gamma^{\frac{R\cdot(R-1)}{2}} \cdot |\boldsymbol{\Sigma}|^{R-T}} \cdot \frac{|\boldsymbol{\Psi}_{TR}(s)|}{\alpha_T(\boldsymbol{\Sigma}) \cdot \alpha_R(\boldsymbol{\Omega}) \cdot \psi_2^{(R)}(-\tilde{s}, R)}, \quad (19)$$

with auxiliary variable

$$\chi = \Gamma_R(R) \cdot (-1)^{\frac{R\cdot(R-1)}{2}} \cdot (-1)^{\frac{(R-T)\cdot(R-T-1)}{2}}, \quad (20)$$

and the R×R matrix

$$\boldsymbol{\Psi}_{TR}(s) = \begin{bmatrix} \boldsymbol{\Psi}_{TR,1}(s) \\ \boldsymbol{\Psi}_{TR,2}(s) \end{bmatrix}, \quad (21)$$

the T×R matrix (i runs from 1 to T and j from 1 to R)

$$\begin{aligned}\boldsymbol{\Psi}_{TR,1}(s) &= \left[{}_2F_0(-\tilde{s}-R+1, 1;\ ;-\gamma\sigma_i\omega_j)\right] \\ &= \left[\frac{1}{\gamma\sigma_i\omega_j} \cdot U\left(1, \tilde{s}+R+1, \frac{1}{\gamma\sigma_i\omega_j}\right)\right]\end{aligned}, \quad (22)$$

and the (R-T)×R matrix (i' runs from 1 to R-T and j' from 1 to R)

$$\boldsymbol{\Psi}_{TR,2}(s) = \left[(-\gamma\omega_{j'})^{i'-1}[-\tilde{s}-R+1]_{i'-1}\right]. \quad (23)$$

Proof: See [20] for details.

B. MGF of transmit correlated channel

Again, we directly start with

Theorem 3. The MGF Z_T of the transmit correlated ($\boldsymbol{\Omega} = \boldsymbol{I}$) MIMO channel mutual information according to (7) with T transmit and R receive antennas is given by

$$Z_T(s) = \frac{(-1)^{\frac{T\cdot(T-1)}{2}}}{|\boldsymbol{\Sigma}|^{R-T+1} \cdot \prod_{k=1}^{T}\Gamma(R-T+k)} \cdot \frac{|\boldsymbol{\Psi}_T(s)|}{\alpha_T(\boldsymbol{\Sigma})} \quad (24)$$

with the T×T matrix (i,j run from 1 to T)

$$\boldsymbol{\Psi}_T(s) = \left[\int_0^\infty e^{-\frac{1}{\sigma_i}t} \cdot t^{R-T+j-1} \cdot (1+\gamma t)^{\tilde{s}} dt\right]. \quad (25)$$

Proof: In case of transmit correlation only, we have to calculate the limit $\boldsymbol{\Omega} \to \boldsymbol{I}$ in (19). To this end, we can apply L'Hospital's rule, i.e. we differentiate nominator and denominator (k-1) times with respect to ω_k, which we symbolically describe by $\frac{\partial}{\partial\boldsymbol{\omega}}$, and then set $\boldsymbol{\omega} = \boldsymbol{1}$. We find

$$\left.\frac{\partial\boldsymbol{\Psi}_{TR,1}}{\partial\boldsymbol{\omega}}\right|_{\boldsymbol{\omega}=\boldsymbol{1}} = \left[\eta_j \cdot (-\gamma\sigma_i)^{j-1} \cdot {}_2F_0(-\tilde{s}-R+j, j;\ ;-\gamma\sigma_i)\right], \quad (26)$$

with auxiliary variable

$$\eta_j = [-\tilde{s}-R+1]_{j-1} \cdot [1]_{j-1}, \quad (27)$$

which can be rewritten in terms of the Kummer U function

$$\left.\frac{\partial\boldsymbol{\Psi}_{TR,1}}{\partial\boldsymbol{\omega}}\right|_{\boldsymbol{\omega}=\boldsymbol{1}} = \left[\eta_j \cdot (-1)^{j-1} \cdot \frac{1}{\gamma\sigma_i} \cdot U\left(j, \tilde{s}+R+1, \frac{1}{\gamma\sigma_i}\right)\right]. \quad (28)$$

On the other hand we get

$$\left.\frac{\partial\boldsymbol{\Psi}_{TR,2}}{\partial\boldsymbol{\omega}}\right|_{\boldsymbol{\omega}=\boldsymbol{1}} = \left[(-1)^{i+j-2} \cdot \gamma^{i-1} \cdot [-\tilde{s}-R+1]_{i-1} \cdot [1-i]_{j-1}\right]. \quad (29)$$

This is a lower triangular matrix. The expression $|\boldsymbol{\Psi}_{TR}|$ can be simplified by developing the determinant of the R×R matrix $\boldsymbol{\Psi}_{TR}$ along the diagonal of the triangular matrix $\boldsymbol{\Psi}_{TR,2}$ and a T×T matrix remains. Without going too much into the details, we can row- and columnwise extract common factors from (28), such that only the Kummer U function remains in the elements of the determinant. Then using the recurrence relation [14], equation 13.4.17

$$U(a, b-1, z) = U(a, b, z) - a \cdot U(a+1, b, z), \quad (30)$$

we can reduce the determinant entries of $|\boldsymbol{\Psi}_{TR}|$ to the following form

$$\begin{aligned} &U\left(j+R-T, \tilde{s}+R-T+j+1, \frac{1}{\gamma\sigma_i}\right) = \\ &\frac{1}{\Gamma(R-T+j)}\int_0^\infty e^{-\frac{1}{\gamma\sigma_i}t} \cdot t^{R-T+j-1} \cdot (1+t)^{\tilde{s}} dt \end{aligned} \quad (31)$$

by iteratively subtracting scaled versions of the (j+1)th column from the jth column. Then note that the integral in (31) can be rewritten as

$$\begin{aligned} &\int_0^\infty e^{-\frac{1}{\gamma\sigma_i}t} \cdot t^{R-T+j-1} \cdot (1+t)^{\tilde{s}} dt = \\ &\gamma^{R-T+j}\int_0^\infty e^{-\frac{1}{\sigma_i}t} \cdot t^{R-T+j-1} \cdot (1+\gamma t)^{\tilde{s}} dt \end{aligned}. \quad (32)$$

Finally, we can find the limit of the Vandermonde determinant

$$\left.\frac{\partial}{\partial\boldsymbol{\omega}}\alpha_R(\boldsymbol{\Omega})\right|_{\boldsymbol{\omega}=\boldsymbol{1}} = (-1)^{\frac{R\cdot(R-1)}{2}} \cdot \Gamma_R(R). \quad (33)$$

Plugging the results in (19) and simplifying yields (24) and (25). *QED.*

We note that with the identity

$$\alpha_T(\boldsymbol{\Sigma}) = |\boldsymbol{\Sigma}|^{T-1} \cdot \alpha_T(\boldsymbol{\Sigma}^{-1}) \cdot (-1)^{\frac{T\cdot(T-1)}{2}} \quad (34)$$

it can be shown that (24) and the result derived in [5] agree.

C. MGF of receive correlated channel

If the receiver side with R>T antennas experiences fading correlation, we get

Theorem 4. The MGF Z_R of the receive correlated MIMO channel ($\boldsymbol{\Sigma} = \boldsymbol{I}$) mutual information according to (7) with T transmit and R receive antennas is given by

$$Z_R(s) = \frac{|\boldsymbol{\Omega}|^{R-T-1} \cdot \begin{vmatrix} \boldsymbol{\Psi}_{R,1}(s) \\ \boldsymbol{\Psi}_{R,2} \end{vmatrix} \cdot (-1)^{\frac{R \cdot (R-1)}{2}} \cdot (-1)^{-(R-T) \cdot T}}{\alpha_R(\boldsymbol{\Omega}) \cdot \Gamma_T(T)}, \quad (35)$$

with the T×R matrix (i runs from 1 to T and j from 1 to R)

$$\boldsymbol{\Psi}_{R,1}(s) = \int_0^\infty e^{-\frac{1}{\omega_j}t} \cdot t^{i-1} \cdot (1+\gamma t)^{\tilde{s}} dt. \quad (36)$$

and the (R-T)×R matrix (i' runs from 1 to (R-T) and j' from 1 to R)

$$\boldsymbol{\Psi}_{R,2} = \omega_{j'}^{i'-(R-T)} = \left(\frac{1}{\omega_{j'}}\right)^{R-T-i'}. \quad (37)$$

Proof: Again we calculate a limit $\boldsymbol{\Sigma} \to \boldsymbol{I}$ of (19) via L'Hospital's rule yielding

$$\left.\frac{\partial \boldsymbol{\Psi}_{TR,1}}{\partial \boldsymbol{\sigma}}\right|_{\boldsymbol{\sigma}=1} = \left[\frac{(-1)^{i-1} \cdot [-\tilde{s}-R+1]_{i-1} \cdot \Gamma(i)}{\gamma\omega_j} \cdot U\left(i, \tilde{s}+R+1, \frac{1}{\gamma\omega_j}\right)\right]. \quad (38)$$

Then consider $|\boldsymbol{\Psi}_{TR}|$. First plug (38) in (21) and factor out $(-1)^{i-1} \cdot [-\tilde{s}-R+1]_{i-1} \cdot \Gamma(i)$ in the first T rows of the resulting determinant. With the formulas

$$U(1,b,z) = \frac{1}{z} + \frac{1}{z}(b-2)U(1,b-1,z), \quad (39)$$

which can be established via integration by parts and in the general case [14] 13.4.18

$$U(a,b,z) = \frac{1}{z} \cdot [(b-a-1) \cdot U(a,b-1,z) + U(a-1,b-1,z)] \quad (40)$$

we can iteratively subtract a scaled version of the ith row from the (i+1)th row and arrive at a determinant, where the first T rows have elements

$$(-1)^{R-T}[-\tilde{s}-R+i]_{R-T} \cdot (\gamma\omega_j)^{R-T-1} \cdot U\left(i, \tilde{s}+T+1, \frac{1}{\gamma\omega_j}\right). \quad (41)$$

Now iteratively add a scaled version of (i+1)th row to the ith row and apply (30), such that we finally find the elements of the first T rows

$$(-1)^{R-T}[-\tilde{s}-R+i]_{R-T} \cdot (\gamma\omega_j)^{R-T-1} \cdot U\left(i, \tilde{s}+i+1, \frac{1}{\gamma\omega_j}\right). \quad (42)$$

Then note that the Kummer U function can explicitly be written as an integral

$$U\left(i, \tilde{s}+i+1, \frac{1}{\gamma\omega_j}\right) = \frac{1}{\Gamma(i)} \cdot \int_0^\infty e^{-\frac{1}{\gamma\omega_j}t} \cdot t^{i-1} \cdot (1+t)^{\tilde{s}} dt. \quad (43)$$

Plugging the results in (19) and simplifying yields (35)-(37). *QED.*

D. MGF of uncorrelated channel

Theorem 5. The MGF Z_U of the uncorrelated MIMO channel ($\boldsymbol{\Sigma} = \boldsymbol{I}, \boldsymbol{\Omega} = \boldsymbol{I}$) mutual information according to (7) with T transmit and R receive antennas is given by

$$Z_U(s) = \frac{|\boldsymbol{\Psi}_U(s)|}{\Gamma_T(T) \cdot \prod_{k=1}^{T} \Gamma(R-T+k)} \quad (44)$$

with the T×T matrix

$$\boldsymbol{\Psi}_U(s) = \left[\int_0^\infty e^{-t} \cdot t^{R-T+i+j-2} \cdot (1+\gamma t)^{\tilde{s}} dt\right], \quad (45)$$

with i and j ranging from 1 to T.

Proof: We can start the derivation with the results of (24) and (25) for a transmit correlated channel. We have to find the limit $\boldsymbol{\Sigma} \to \boldsymbol{I}$, which can be obtained by L'Hospital's rule. By Lebesgue's dominated convergence theorem we can exchange the sequence of integration and differentiation in (25), where we can use

$$\frac{\partial^k}{\partial c^k} e^{-t/c} = \left(\sum_{l=1}^{k-1} \frac{a_l}{c^{l+k}} \cdot t^l + \frac{t^k}{c^{2k}}\right) \cdot e^{-t/c}, \quad (46)$$

where the a_l are constants. Adding a properly scaled multiple of the 1st, 2nd, …, (i-1)th row to the ith row in the determinant resulting from the differentiation of (25) and using (33) for differentiating the denominator yields finally (44) and (45). *QED.*

5. Calculation of Ergodic Capacity

Note that from the MGF of mutual information we can derive the ergodic capacity $C_{erg,x}$ with uninformed transmitter, which is the first moment of mutual information

$$C_{\mathrm{erg},x} = E_{\boldsymbol{H}_w}[I(s,y)] = E_{\boldsymbol{H}_w}[\log_2|\boldsymbol{I} + \gamma \cdot \boldsymbol{\Sigma}\boldsymbol{H}_w^H\boldsymbol{\Omega}\boldsymbol{H}_w|] \quad (47)$$

by

$$C_{\mathrm{erg},x} = \left.\frac{d}{ds} Z_x(s)\right|_{s=0}. \quad (48)$$

In the subsequent calculations, we can make use of the following formula for the derivative of a determinant, which can be established via the product rule of differentiation

$$\frac{\partial}{\partial s}|\boldsymbol{X}(s)| = \sum_i |\boldsymbol{X}_i(s)|, \quad (49)$$

where $|\boldsymbol{X}_i(s)|$ is the determinant of a general matrix **X**, where the ith column (or alternatively row) is differentiated with respect to s.

The following theorems are given without proof, however, the interested reader can derive them via (48) together with (49) and noting, that by Lebesgue's dominated convergence theorem we can exchange the sequence of integration and differentiation.

A. Fully correlated channel

Theorem 6. The ergodic capacity according to (7) of a fully correlated T×R MIMO system is given by

$$C_{erg,TR} = \frac{\Gamma_R(R) \cdot (-1)^{\frac{(R-T)\cdot(R-T-1)}{2}}}{\ln 2 \cdot \gamma^{\frac{R\cdot(R-1)}{2}} \cdot \prod_{k=1}^{R-1} k^R} \cdot \frac{\sum_{l=1}^{T} \begin{vmatrix} \Xi_{TR}(l) \\ \boldsymbol{\Psi}_{TR,2}(0) \end{vmatrix}}{\alpha_T(\boldsymbol{\Sigma}) \cdot \alpha_R(\boldsymbol{\Omega})} \quad (50)$$

with the T×R matrices (i runs from 1 to T and j from 1 to R)

$$\Xi_{TR}(l) = \begin{bmatrix} \Gamma(R)\cdot\sigma_i^{T-1}\cdot(\gamma\omega_j)^{R-1}\cdot e^{\frac{1}{\gamma\sigma_i\omega_j}}\cdot \mathrm{E}_1\left(\frac{1}{\gamma\sigma_i\omega_j}\right) & i = l \\ \sum_{k=R-T}^{R-1}(-1)^k\cdot(\sigma_i)^{k-(R-T)}\cdot(\gamma\omega_j)^k\cdot[1-R]_k & i \neq l \end{bmatrix}, \quad (51)$$

where the exponential integral is defined by [14] 5.1.1

$$\mathrm{E}_1(x) = \int_x^{\infty}\frac{e^{-t}}{t}dt \quad (52)$$

and $\boldsymbol{\Psi}_{TR,2}(s)$ given in (23).

Proof: See [20] for details.

B. Transmit correlated channel

Theorem 7. The ergodic capacity of a transmit correlated T×R MIMO system ($\boldsymbol{\Omega} = \boldsymbol{I}$) according to (7) is given by

$$C_{\mathrm{erg},T} = \frac{(-1)^{\frac{T\cdot(T-1)}{2}}}{\ln 2\cdot|\boldsymbol{\Sigma}|^{R-T+1}\cdot\alpha_T(\boldsymbol{\Sigma})}\cdot\sum_{l=1}^{T}\left|\Xi_T(l)\right| \quad (53)$$

with the T×T matrices (i,j run from 1 to T)

$$\Xi_T(l) = \begin{bmatrix} \left(\begin{array}{l}\frac{1}{\Gamma(R-T+j)}\int_0^{\infty}e^{-\frac{1}{\sigma_i}t}\cdot t^{R-T+j-1}\cdot\ln(1+\gamma t)dt \\ = \frac{1}{\Gamma(R-T+j)}\cdot I_1\left(\frac{1}{\sigma_i},\gamma,R-T+j\right)\end{array}\right) & i = l \\ \frac{1}{\Gamma(R-T+j)}\int_0^{\infty}e^{-\frac{1}{\sigma_i}t}\cdot t^{R-T+j-1}dt = \sigma_i^{R-T+j} & i \neq l \end{bmatrix}. \quad (54)$$

Note that via integration by parts one can derive a closed form of the integral I_1 in (54) [see e.g. [13] equation (78)]

$$\begin{aligned} \mathrm{I}_1(c,a,n) &= \int_0^{\infty}e^{-ct}\cdot t^{n-1}\cdot\ln(1+at)dt \\ &= \Gamma(n)\cdot e^{c/a}\cdot\sum_{k=1}^{n}\frac{\Gamma\left(-n+k,\frac{c}{a}\right)}{c^k\cdot a^{n-k}} \end{aligned}. \quad (55)$$

Moreover, we have from [14] 6.5.19 the special case of the incomplete Gamma function

$$\Gamma(-n,x) = \frac{(-1)^n}{n!}\cdot\left[E_1(x)-e^{-x}\cdot\sum_{j=0}^{n-1}\frac{(-1)^j\cdot j!}{x^{j+1}}\right]. \quad (56)$$

C. Receive correlated channel

Theorem 8. The ergodic capacity of a receive correlated T×R MIMO system ($\boldsymbol{\Sigma} = \boldsymbol{I}$) is given by

$$C_{\mathrm{erg},R} = \frac{(-1)^{\frac{R\cdot(R-1)}{2}}\cdot(-1)^{-(R-T)\cdot T}\cdot|\boldsymbol{\Omega}|^{R-T-1}}{\ln 2\cdot\alpha_R(\boldsymbol{\Omega})\cdot\Gamma_T(T)}\cdot\sum_{l=1}^{T}\left|\begin{matrix}\Xi_R(l)\\ \boldsymbol{\Psi}_{R,2}\end{matrix}\right|, \quad (57)$$

with the T×R matrices (i runs from 1 to T, j runs from 1 to R)

$$\Xi_R(l) = \begin{bmatrix} \int_0^{\infty}e^{-\frac{1}{\omega_j}t}\cdot t^{i-1}\cdot\ln(1+\gamma t)dt = \mathrm{I}_1\left(\frac{1}{\omega_j},\gamma,i\right) & i = l \\ \int_0^{\infty}e^{-\frac{1}{\omega_j}t}\cdot t^{i-1}dt = \Gamma(i)\cdot\omega_j^i & i \neq l \end{bmatrix}, \quad (58)$$

the (R-T)×R matrix (i' runs from 1 to (R-T), j' runs from 1 to R)

$$\boldsymbol{\Psi}_{R,2} = \left(\frac{1}{\omega_{j'}}\right)^{R-T-i'} \quad (59)$$

and the integral $I_1(c,a,n)$ in (55).

D. Uncorrelated channel

Theorem 9. The ergodic capacity of an uncorrelated T×R MIMO system ($\boldsymbol{\Sigma} = \boldsymbol{I}, \boldsymbol{\Omega} = \boldsymbol{I}$) is given by

$$C_{\mathrm{erg,U}} = \frac{1}{\ln 2\cdot\Gamma_T(T)\cdot\prod_{k=1}^{T}\Gamma(R-T+k)}\cdot\sum_{l=1}^{T}\left|\Xi_U(l)\right| \quad (60)$$

with the T×T matrices (i,j run from 1 to T)

$$\Xi_U(l) = \begin{bmatrix} \left(\begin{array}{l}\int_0^{\infty}e^{-t}\cdot t^{R-T+i+j-2}\cdot\ln(1+\gamma t)dt \\ = I_1(1,\gamma,R-T+i+j-1)\end{array}\right) & i = l \\ \int_0^{\infty}e^{-t}\cdot t^{R-T+i+j-2}dt = \Gamma(R-T+i+j-1) & i \neq l \end{bmatrix} \quad (61)$$

and the integral $I_1(c,a,n)$ in (55).

6. Simulation Results

Without loss of generality, we study systems with white input signals of power E_s and additive white Gaussian noise of variance N_0, i.e.

$$\boldsymbol{R}_{ss} = E_s\cdot\boldsymbol{I} \qquad \boldsymbol{R}_{nn} = N_0\cdot\boldsymbol{I}. \quad (62)$$

Following [16], we assume a single main direction of departure and arrival, respectively, of 20 degrees with respect to the array perpendicular at RX and TX. The array element spacing is 0.5 wavelengths. Furthermore, there is a Laplacian power distribution over the angular spread (AS). $\mathbf{R}_{RX}$ and $\mathbf{R}_{TX}$ are determined according to these assumptions. The SNR in dB is defined by

$$SNR = 10\cdot\log_{10}\left(\frac{T\cdot E_s}{N_0}\right) = 10\cdot\log_{10}\left(\frac{\rho}{N_0}\right) \quad [dB], \quad (63)$$

where ρ is the total transmitted power per channel use.

Simulation results and theoretical curves closely agree in Fig. 1 for a 6×6 system with fully correlated (correlation at both transmitter and receiver) MIMO channel. As expected, the negative impact of channel correlation on ergodic capacity with uninformed transmitter can be observed.

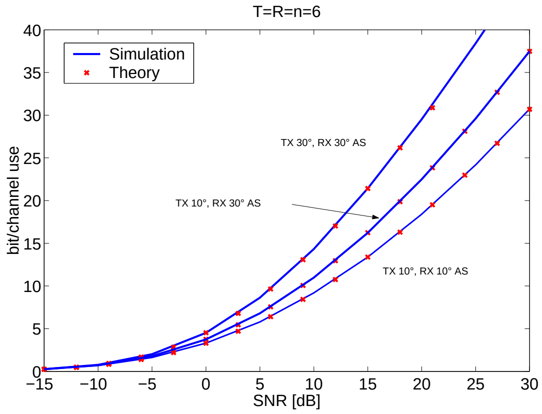

Fig. 1: Ergodic capacity of fully correlated MIMO system (T=R=6)

To give an impression of the impact of the propagation scenario on the eigenvalues of the correlation matrices (the decisive parameters that determine ergodic capacity), note that for 10° and 30° AS in the 6×6 case we find

$$\begin{aligned} \mathrm{eig}(\boldsymbol{R}_{10^\circ}) &= \begin{bmatrix} 4.1460 & 1.3132 & 0.4020 & 0.1097 & 0.0252 & 0.0039 \end{bmatrix} \\ \mathrm{eig}(\boldsymbol{R}_{30^\circ}) &= \begin{bmatrix} 2.1721 & 1.4616 & 1.0278 & 0.7075 & 0.4105 & 0.2205 \end{bmatrix} \end{aligned}. \tag{64}$$

At this point we emphasize that it can be shown that the ergodic MIMO capacity is a Schur-concave function of the eigenvalues of the transmit as well as the receive correlation matrix. Loosely speaking, a higher spread of the eigenvalues leads to lower ergodic capacity. Moreover, note in Fig. 1 that the capacity loss due to increased transmit correlation (difference between upper and middle curve) is greater than the loss induced by additional receive correlation (difference between middle and lower curve).

For a fully correlated system with 10° AS at RX and TX we have depicted the dependence of ergodic capacity on the number of transmit antennas in Fig. 2. Obviously, T=1 corresponds to the receive diversity case in standard single input multiple output (SIMO) smart antenna systems. As expected, there is a great gain in going from 1 to 2 transmit antennas, while the gain gets smaller if we add more and more transmit antennas. A similar effect can be observed when considering SER curves of diversity systems. Specifically, in the limit of a large number of transmit antennas it can be shown that ergodic capacity saturates at a fixed value which solely depends on the transmitted power and the noise power. The interested reader is referred to [17], where asymptotical results for uncorrelated MIMO channels can be found.

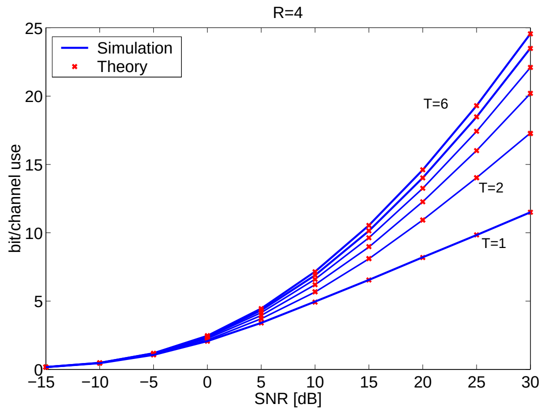

Fig. 2: Ergodic capacity, 10° AS at RX and TX (R=4)

In Fig. 3, the lower 3 curves results from a system simulation with T=2 transmit antennas and R=2, 4, and 6 receive antennas without fading correlation. Similar results are given for a system with T=4 and R=4,6,8,10 in the upper 4 curves. An asymptotic analysis in the limit of a large number of receive antennas reveals that ergodic capacity increases with the logarithm of the number of antennas. Again, for the uncorrelated case these asymptotics are given in [17]. An extension to cover correlated MIMO systems is straightforward.

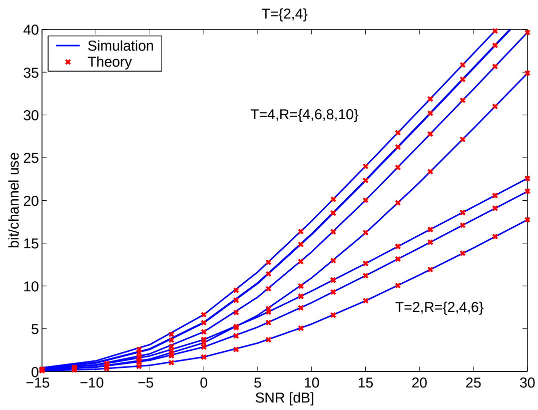

Fig. 3: Ergodic capacity, uncorrelated, T=2, T=4, various R

In Fig. 4 we present theoretical results on the ergodic capacity for a T=4, R=4 system with uncorrelated fading at the transmitter and correlated receiver for different SNRs, which is varied in 5 dB steps from -10 dB to 20 dB. Obviously, there is a significant reduction of capacity with highly correlated channels, i.e. for low angular spreads. This effect is especially pronounced for higher SNR, while at low SNR correlation basically has no effect on capacity. These observations can be underpinned by theoretical studies [21].

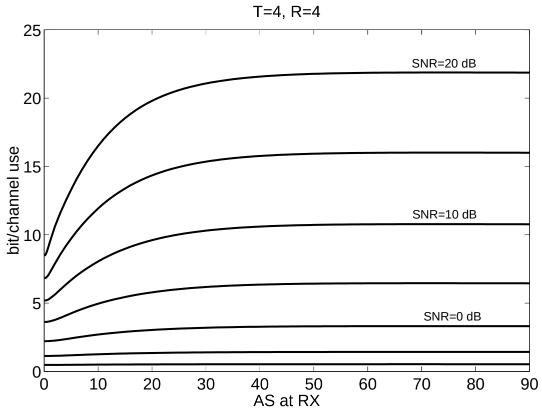

Fig. 4: Ergodic capacity, T=4, R=4, TX unc., various AS at RX

In contrast to the results for the MIMO system in Fig. 4, the influence of fading correlation is not as severe for a SIMO system in Fig. 5 with T=1 transmit antenna and R=7 receive antennas. Low angular spreads at the receiver lead only to a minor performance degradation, even at higher SNR.

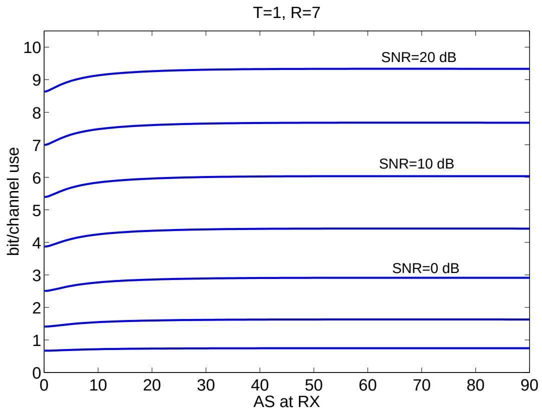

Fig. 5: Ergodic capacity, T=1, R=7, TX unc., various AS at RX

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